

NON-LINEAR *-JORDAN DERIVATIONS ON VON NEUMANN ALGEBRAS

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ABSTRACT. Let \mathcal{A} be a factor von Neumann algebra and ϕ be the *-Jordan derivation on \mathcal{A} , that is, for every $A, B \in \mathcal{A}$, $\phi(A \diamond_1 B) = \phi(A) \diamond_1 B + A \diamond_1 \phi(B)$ where $A \diamond_1 B = AB + BA^*$, then ϕ is additive *-derivation.

1. INTRODUCTION

Let \mathcal{R} and \mathcal{R}' be rings. We say the map $\Phi : \mathcal{R} \rightarrow \mathcal{R}'$ preserves product or is multiplicative if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{R}$. The question of when a product preserving or multiplicative map is additive was discussed by several authors, see [16] and references therein. Motivated by this, many authors pay more attention to the map on rings (and algebras) preserving Lie product $[A, B] = AB - BA$ or Jordan product $A \circ B = AB + BA$ (for example, see [1, 2, 5, 8, 12, 13, 15, 19]). These results show that, in some sense, Jordan product or Lie product structure is enough to determine the ring or algebraic structure. Historically, many mathematicians devoted themselves to the study of additive or linear Jordan or Lie product preservers between rings or operator algebras. Such maps are always called Jordan homomorphism or Lie homomorphism. Here we only list several results [6, 7, 9, 16, 17, 18].

Let \mathcal{R} be a *-ring. For $A, B \in \mathcal{R}$, denoted by $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, which are *-Jordan product and *-Lie product, respectively. These products are found playing a more and more important role in some research topics, and its study has recently attracted many author's attention (for example, see [14, 20, 4, 11]).

Let define ξ -Jordan *-product by $A \diamond_\xi B = AB + \xi BA^*$. We say the map ϕ with property of $\phi(A \diamond_\xi B) = \phi(A) \diamond_\xi B + A \diamond_\xi \phi(B)$ is a ξ -Jordan *-derivation map. It is clear that for $\xi = -1$ and $\xi = 1$, the ξ -Jordan *-derivation map is a *-Lie derivation and *-Jordan derivation, respectively [3]. We should mention here whenever we say ϕ preserves derivation, it means $\phi(AB) = \phi(A)B + A\phi(B)$. Recently, Yu and Zhang in [21] proved that every non-linear *-Lie derivation from a factor von Neumann algebra into itself is an additive *-derivation. Also, Li,

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Lu and Fang in [10] have investigated a non-linear ξ -Jordan $*$ -derivation. They showed that if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra without central abelian projections and ξ is a non-zero scalar, then $\phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a non-linear ξ -Jordan $*$ -derivation if and only if ϕ is an additive $*$ -derivation.

Let \mathcal{H} be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be all bounded linear operators on \mathcal{H} . In this paper we show that $*$ -Jordan derivation map on every factor von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is additive $*$ -derivation.

Note that a subalgebra \mathcal{A} of $\mathcal{B}(\mathcal{H})$ is called a von Neumann algebra when it is closed in the weak topology of operators. A von Neumann algebra \mathcal{A} is called factor when its center is trivial. It is clear that if \mathcal{A} is a factor von Neumann algebra, then \mathcal{A} is prime, that is, for $A, B \in \mathcal{A}$ if $AAB = \{0\}$, then $A = 0$ or $B = 0$. We denote real and imaginary part of an operator A by $\Re(A)$ and $\Im(A)$, respectively i.e., $\Re(A) = \frac{A+A^*}{2}$ and $\Im(A) = \frac{A-A^*}{2i}$.

2. THE STATEMENT OF THE MAIN THEOREM

The statement of our main theorem is the following.

Main Theorem. Let \mathcal{A} be a factor von Neumann algebra acting on complex Hilbert space \mathcal{H} and $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ be a $*$ -Jordan derivation on \mathcal{A} , that is, for every $A, B \in \mathcal{A}$

$$\phi(A \diamond_1 B) = \phi(A) \diamond_1 B + A \diamond_1 \phi(B) \quad (2.1)$$

where $A \diamond_1 B = AB + BA^*$, then ϕ is additive $*$ -derivation.

Before proving the Main Theorem, we need two lemmas.

Lemma 2.1. *Let $A \in \mathcal{A}$. Then $AB = -BA^*$ for every $B \in \mathcal{A}$ implies that $A \in \mathbb{C}I$.*

Proof. Let $B = I$. We have $A = -A^*$ and thus $AB = BA$ for every $B \in \mathcal{A}$. Therefore, $A \in \mathbb{C}I$, as \mathcal{A} is factor. \square

Let $P_1 \in \mathcal{A}$ be a non-trivial projection and $P_2 = I - P_1$. Let $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$ for $i, j = 1, 2$, we can write $\mathcal{A} = \sum_{i,j=1,2} \mathcal{A}_{ij}$ such that their pairwise intersections are $\{0\}$.

In the following Lemma we use the same idea of [21].

Lemma 2.2. *Let $A \in \mathcal{A}$. Then $AB = -BA^*$ for every $B \in \mathcal{A}_{12}$ implies that there exists $\lambda \in \mathbb{C}$ such that $A = \lambda P_1 - \bar{\lambda} P_2$.*

Proof. We write $A = A_{11} + A_{12} + A_{21} + A_{22}$. From $AB = -BA^*$ we have $(A_{11} + A_{12} + A_{21} + A_{22})B = -B(A_{11}^* + A_{12}^* + A_{21}^* + A_{22}^*)$. Hence, $A_{11}B + A_{21}B = -BA_{12}^* - BA_{22}^*$, for $B \in \mathcal{A}_{12}$. Multiplying the latter equation by P_2 from the left side, implies that $A_{21}B = 0$ and therefore,

$$A_{21} = 0. \quad (2.2)$$

as \mathcal{A} is prime.

For every $X \in \mathcal{A}_{11}$ and $B \in \mathcal{A}_{12}$, we can write $AXB = -XBA^*$ and also $XAB = -XBA^*$ since XB is in \mathcal{A}_{12} . Hence, $(AX - XA)B = 0$ so, $(AX - XA)P_1TP_2 = 0$ for every $T \in \mathcal{A}$. Thus, $(AX - XA)P_1 = 0$ because of primeness, so we can write $P_1AP_1X = XP_1AP_1$ since $X \in \mathcal{A}_{12}$ and $A \in \mathcal{A}$. Therefore, there exists $\lambda \in \mathbb{C}$ such that

$$P_1AP_1 = \lambda P_1, \quad (2.3)$$

as \mathcal{A} is factor.

For every $Y \in \mathcal{A}_{22}$ and $B \in \mathcal{A}_{12}$, we can write $ABY = -BYA^*$ and also $ABY = -BA^*Y$, since BY is in \mathcal{A}_{12} .

By a similar way, we can obtain

$$P_2AP_2 = \mu P_2, \quad (2.4)$$

for some $\mu \in \mathbb{C}$.

Let $A = P_1AP_1 + P_1AP_2 + P_2AP_1 + P_2AP_2$, Equations (2.2), (2.3) and (2.4) imply that

$$A = \lambda P_1 + \mu P_2 + P_1AP_2. \quad (2.5)$$

Also, From $B \in \mathcal{A}_{12}$ and Equation (2.3) we can write $P_1AP_1B = \lambda P_1B = \lambda B$, it follows $P_1AB = \lambda B$. From the latter Equation and (2.5) we have

$$\lambda B = P_1AB = -BA^* = -\bar{\mu}B - BA^*P_1.$$

Multiplying above equation by P_2 from the right side, we have $\lambda B = -\bar{\mu}B$ for every $B \in \mathcal{A}_{12}$. It follows, $\mu = -\bar{\lambda}$ and so, $BA^*P_1 = 0$ or $BP_2A^*P_1 = 0$ for every $B \in \mathcal{A}_{12}$. Hence, $P_2A^*P_1 = 0$ or $P_1AP_2 = 0$. By Equation (2.5), we obtain $A = \lambda P_1 - \bar{\lambda}P_2$, where $\lambda \in \mathbb{C}$. This completes the proof of Lemma. \square

Now we prove our Main Theorem in several Steps.

Step 1. $\phi(0) = 0$ and $\phi(P_i)$ are self-adjoint for $i = 1, 2$.

By Equation (2.1), it is easy to obtain $\phi(0) = 0$.

Now, we prove that $\phi(P_i)$ are self-adjoint for $i = 1, 2$. Let A be a self-adjoint operator in \mathcal{A} . Since $A \diamond_1 P_i = P_i \diamond_1 A$, we can write $\phi(A \diamond_1 P_i) = \phi(P_i \diamond_1 A)$. Then, by Equation (2.1) we have

$$\phi(A)P_i + P_i\phi(A)^* + A\phi(P_i) + \phi(P_i)A = \phi(P_i)A + A\phi(P_i)^* + P_i\phi(A) + \phi(A)P_i,$$

or,

$$A(\phi(P_i) - \phi(P_i)^*) = P_i(\phi(A) - \phi(A)^*).$$

We multiply above equation by P_j from left side, it follows

$$P_jA(\phi(P_i) - \phi(P_i)^*) = 0,$$

for all $A \in \mathcal{A}$. It means $P_j\mathcal{A}(\phi(P_i) - \phi(P_i)^*) = \{0\}$. So, we have $\phi(P_i) = \phi(P_i)^*$, for $i = 1, 2$ by primeness property of \mathcal{A} .

Step 2. Let $U = P_1\phi(P_1)P_2 - P_2\phi(P_1)P_1$, we have

- (a) for every $A \in \mathcal{A}_{12}$, $\phi(A) = AU - UA + P_1\phi(A)P_2$
- (b) for every $B \in \mathcal{A}_{21}$, $\phi(B) = BU - UB + P_2\phi(B)P_1$
- (c) there exist $\alpha_i \in \mathbb{C}$ such that $\phi(P_i) = P_iU - UP_i + \alpha_i P_i$ for every $i = 1, 2$.

(a) Let $A \in \mathcal{A}_{12}$, we can obtain $A = P_1 \diamond_1 A$, by Step 1 we have

$$\phi(A) = \phi(P_1)A + A\phi(P_1) + P_1\phi(A) + \phi(A)P_1.$$

Multiplying above equation by P_1 and P_2 from two sides, respectively, we consider the following equation have

$$\begin{aligned} P_1\phi(A)P_1 &= -A\phi(P_1)P_1, \\ P_2\phi(A)P_2 &= P_2\phi(P_1)A, \\ P_1\phi(P_1)A &= -A\phi(P_1)P_2. \end{aligned} \tag{2.6}$$

On the other hand, from $P_1 \diamond_1 P_2 = 0$ we have

$$\phi(P_1)P_2 + P_2\phi(P_1) + P_1\phi(P_2) + \phi(P_2)P_1 = 0.$$

Hence, multiplying above equation by P_2 from right and left side, we have

$$P_2\phi(P_1)P_2 = 0. \tag{2.7}$$

Since $A \diamond_1 P_1 = 0$, for $A \in \mathcal{A}_{12}$, we have $\phi(A)P_1 + P_1\phi(A)^* + A\phi(P_1) + \phi(P_1)A^* = 0$. Multiplying the latter equation by P_2 from the left side, it is clear that

$$P_2\phi(A)P_1 + P_2\phi(P_1)A^* = 0$$

and it follows

$$P_2\phi(A)P_1 + P_2\phi(P_1)P_2A^*P_1 = 0,$$

since $A \in \mathcal{A}_{12}$. The Equation (2.7) shows $P_2\phi(A)P_1 = 0$. Hence, by assumption of U and Equation (2.6) we have

$$\begin{aligned} \phi(A) &= P_2\phi(A)P_2 + P_1\phi(A)P_1 + P_2\phi(A)P_1 + P_1\phi(A)P_2 \\ &= P_2\phi(P_1)A - A\phi(P_1)P_1 + P_1\phi(A)P_2 \\ &= AU - UA + P_1\phi(A)P_2. \end{aligned}$$

(b) Let $B \in \mathcal{A}_{21}$. From $B = P_2 \diamond_1 B$ and similar to (a) we can obtain $P_2\phi(B)P_2 = -B\phi(P_2)P_2$, $P_1\phi(B)P_1 = P_1\phi(P_2)B$ and $P_1\phi(B)P_2 = 0$.

Hence

$$\begin{aligned} \phi(B) &= P_1\phi(B)P_1 + P_2\phi(B)P_2 + P_2\phi(B)P_1 + P_1\phi(B)P_2 \\ &= P_1\phi(P_2)B - B\phi(P_2)P_2 + P_2\phi(B)P_1. \end{aligned}$$

On the other hand, from relation $P_2 \diamond_1 P_1 = 0$ we can obtain

$$P_1\phi(P_2)P_2 = -P_1\phi(P_1)P_2.$$

Multiplying above equation by B from two sides. We have $B\phi(P_2)P_2 = -B\phi(P_1)P_2$ and $P_1\phi(P_2)B = -P_1\phi(P_1)B$. Therefore,

$$\phi(B) = B\phi(P_1)P_2 - P_1\phi(P_1)B + P_2\phi(A)P_1$$

and from assumption of U we have

$$\phi(B) = BU - UB + P_2\phi(B)P_1.$$

(c) For every $X \in \mathcal{A}_{11}$ and $A \in \mathcal{A}_{12}$ and from relation (2.6) we can write $P_1\phi(P_1)XA = -XA\phi(P_1)P_2$ and $XP_1\phi(P_1)A = -XA\phi(P_1)P_2$. Therefore, $[P_1\phi(P_1)X - XP_1\phi(P_1)]A = 0$ and so $[P_1\phi(P_1)X - XP_1\phi(P_1)P_1]\mathcal{A}P_2 = \{0\}$, for $A \in \mathcal{A}_{12}$. By the primeness of \mathcal{A} and $X \in \mathcal{A}_{11}$, it is clear that $P_1\phi(P_1)P_1X = XP_1\phi(P_1)P_1$.

Since \mathcal{A} is factor, $P_1\phi(P_1)P_1 = \alpha_1 P_1$ for some $\alpha_1 \in \mathbb{C}$. Hence, from Equation (2.7) we can write

$$\phi(P_1) = P_1\phi(P_1)P_2 + P_2\phi(P_1)P_1 + P_1\phi(P_1)P_1 + P_2\phi(P_1)P_2 = P_1U - UP_1 + \alpha_1 P_1.$$

Similar to this way, we can obtain $\phi(P_2) = P_2U - UP_2 + \alpha_2 P_2$ for some $\alpha_2 \in \mathbb{C}$.

Remark 2.3. Let $\psi(X) = \phi(X) - (XU - UX)$ for all $X \in \mathcal{A}$. By a calculation, we can show that ψ is *-Jordan derivation and so, by previous Steps, $\psi(P_i)$ are self-adjoint and

$$\psi(P_i) = \alpha_i P_i \quad (2.8)$$

for $i = 1, 2$. Also, this shows that α_i are real.

Step 3. By assumption of ψ , for every $i, j = 1, 2$, we have $\psi(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$.

Let $i \neq j$, Step 2 and Remark 2.3 show that $\psi(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$.

Let $X \in \mathcal{A}_{ii}$, for $i = 1, 2$. We have $P_j \diamond_1 X = 0$, so

$$\psi(P_j)X + X\psi(P_j) + P_j\psi(X) + \psi(X)P_j = 0,$$

since $\psi(P_j) \in \mathcal{A}_{jj}$, by Equation (2.8). Therefore, $P_j\psi(X) + \psi(X)P_j = 0$. Then, by multiplying the latter equation by P_i from right and left side respectively, and P_j from both side, we have $P_i\psi(X)P_j = P_j\psi(X)P_i = P_j\psi(X)P_j = 0$. Hence $\psi(X) \in \mathcal{A}_{ii}$.

Step 4. For $i, j \in \{1, 2\}$ with $i \neq j$, we have

- (a) $\psi(A_{ii} + A_{jj}) = \psi(A_{ii}) + \psi(A_{jj})$
- (b) $\psi(A_{ii} + A_{ij}) = \psi(A_{ii}) + \psi(A_{ij})$
- (c) $\psi(A_{ii} + A_{ji}) = \psi(A_{ii}) + \psi(A_{ji})$
- (d) $\psi(A_{ij} + A_{ji}) = \psi(A_{ij}) + \psi(A_{ji})$.

(a) From $P_i \diamond_1 (A_{ii} + A_{jj}) = P_i \diamond_1 A_{ii}$ we have

$$\psi(P_i) \diamond_1 (A_{ii} + A_{jj}) + P_i \diamond_1 \psi(A_{ii} + A_{jj}) = \psi(P_i) \diamond_1 A_{ii} + P_i \diamond_1 \psi(A_{ii}),$$

So,

$$\psi(P_i) \diamond_1 A_{ii} + \psi(P_i) \diamond_1 A_{jj} + P_i \diamond_1 \psi(A_{ii} + A_{jj}) = \psi(P_i) \diamond_1 A_{ii} + P_i \diamond_1 \psi(A_{ii}).$$

Since $\psi(P_i)$ is a real multiple of P_i , by Equation (2.8). Above equation can be written as

$$P_i \diamond_1 \psi(A_{ii} + A_{jj}) = P_i \diamond_1 \psi(A_{ii}).$$

Hence, $P_i \diamond_1 K = 0$ where $K = \psi(A_{ii} + A_{jj}) - \psi(A_{ii})$. This implies that $P_i K + K P_i = 0$ and so,

$$P_i K P_i = P_i K P_j = P_j K P_i = 0.$$

Therefore, from $\psi(A_{ii}) \in \mathcal{A}_{ii}$ we have

$$P_i \psi(A_{ii} + A_{jj}) P_i = \psi(A_{ii}), \quad (2.9)$$

and

$$P_i \psi(A_{ii} + A_{jj}) P_j = P_j \psi(A_{ii} + A_{jj}) P_i = 0. \quad (2.10)$$

A similar method shows

$$P_j \psi(A_{ii} + A_{jj}) P_j = \psi(A_{jj}), \quad (2.11)$$

since $P_j \diamond_1 (A_{ii} + A_{jj}) = P_j \diamond_1 A_{jj}$.

On the other hand, we can write

$$\psi(A_{ii} + A_{jj}) = P_i \psi(A_{ii} + A_{jj}) P_i + P_i \psi(A_{ii} + A_{jj}) P_j + P_j \psi(A_{ii} + A_{jj}) P_i + P_j \psi(A_{ii} + A_{jj}) P_j$$

So, by Equations (2.9), (2.10) and (2.11) we have the following

$$\psi(A_{ii} + A_{jj}) = \psi(A_{ii}) + \psi(A_{jj}).$$

(b) From $P_j \diamond_1 (A_{ii} + A_{ij}) = P_j \diamond_1 A_{ij}$ we can write

$$\psi(P_j) \diamond_1 (A_{ii} + A_{ij}) + P_j \diamond_1 \psi(A_{ii} + A_{ij}) = \psi(P_j) \diamond_1 A_{ij} + P_j \diamond_1 \psi(A_{ij}).$$

Let $K = \psi(A_{ii} + A_{ij}) - \psi(A_{ij})$. Since $\psi(P_j)$ is a real multiple of P_j , we can write $P_j \diamond_1 K = 0$. Thus, $P_j K + K P_j = 0$ and so

$$P_i K P_j = P_j K P_i = P_j K P_j = 0.$$

Therefore,

$$P_i \psi(A_{ii} + A_{ij}) P_j = \psi(A_{ij}) \quad (2.12)$$

and

$$P_j \psi(A_{ii} + A_{ij}) P_i = P_j \psi(A_{ii} + A_{ij}) P_j = 0. \quad (2.13)$$

On the other hand, $(A_{ii} + A_{ij}) \diamond_1 X_{ii} = A_{ii} \diamond_1 X_{ii}$ for every $X_{ii} \in \mathcal{A}_{ii}$. Hence, $\psi[(A_{ii} + A_{ij}) \diamond_1 X_{ii}] = \psi(A_{ii} \diamond_1 X_{ii})$ and so

$$\psi(A_{ii} + A_{ij}) \diamond_1 X_{ii} + (A_{ii} + A_{ij}) \diamond_1 \psi(X_{ii}) = \psi(A_{ii}) \diamond_1 X_{ii} + A_{ii} \diamond_1 \psi(X_{ii}).$$

This shows $L \diamond_1 X_{ii} = 0$ where $L = \psi(A_{ii} + A_{ij}) - \psi(A_{ii})$. Thus, $L X_{ii} = -X_{ii} L^*$ and from Lemma 2.1. We have $P_i L P_i = \lambda P_i$ for some $\lambda \in \mathbb{C}$. This means

$$P_i \psi(A_{ii} + A_{ij}) P_i = \psi(A_{ii}) + \lambda P_i. \quad (2.14)$$

Therefore, by Equations (2.12), (2.13) and (2.14), we have

$$\psi(A_{ii} + A_{ij}) = \psi(A_{ii}) + \psi(A_{ij}) + \lambda P_i.$$

By applying this method, there exists $\alpha \in \mathbb{C}$ such that

$$\begin{aligned} \psi[(A_{ii} + A_{ij}) \diamond_1 X_{ij}] &= \psi(A_{ii}X_{ij} + X_{ij}A_{ij}^*) \\ &= \psi(A_{ii}X_{ij}) + \psi(X_{ij}A_{ij}^*) + \alpha P_i, \end{aligned}$$

for every $X_{ij} \in \mathcal{A}_{ij}$. Also

$$\begin{aligned} \psi[(A_{ii} + A_{ij}) \diamond_1 X_{ij}] &= \psi(A_{ii} + A_{ij}) \diamond_1 X_{ij} + (A_{ii} + A_{ij}) \diamond_1 \psi(X_{ij}) \\ &= [\psi(A_{ii}) + \psi(A_{ij}) + \lambda P_i] \diamond_1 X_{ij} + A_{ii} \diamond_1 \psi(X_{ij}) + A_{ij} \diamond_1 \psi(X_{ij}) \\ &= \psi(A_{ii} \diamond_1 X_{ij}) + \psi(A_{ij} \diamond_1 X_{ij}) + \lambda P_i \diamond_1 X_{ij} \\ &= \psi(A_{ii}X_{ij}) + \psi(X_{ij}A_{ij}^*) + \lambda X_{ij}. \end{aligned}$$

Then $\alpha P_i = \lambda X_{ij}$ and so $\alpha P_i P_i = \lambda X_{ij} P_i = 0$. So, $\alpha = 0$ and $\lambda = 0$.

This implies that $\psi(A_{ii} + A_{ij}) = \psi(A_{ii}) + \psi(A_{ij})$.

(c) Let $X_{ji} \in \mathcal{A}_{ji}$, then,

$$\psi[(A_{ii} + A_{ji}) \diamond_1 X_{ji}] = \psi(A_{ii} + A_{ji}) \diamond_1 X_{ji} + (A_{ii} + A_{ji}) \diamond_1 \psi(X_{ji}).$$

On the other hand, it follows from (a)

$$\begin{aligned} \psi[(A_{ii} + A_{ji}) \diamond_1 X_{ji}] &= \psi(X_{ji}A_{ii}^* + X_{ji}A_{ji}^*) \\ &= \psi(X_{ji}A_{ii}^*) + \psi(X_{ji}A_{ji}^*) \\ &= \psi(A_{ii} \diamond_1 X_{ji}) + \psi(A_{ji} \diamond_1 X_{ji}) \\ &= \psi(A_{ii}) \diamond_1 X_{ji} + A_{ii} \diamond_1 \psi(X_{ji}) + \psi(A_{ji}) \diamond_1 X_{ji} + A_{ji} \diamond_1 \psi(X_{ji}) \\ &= [\psi(A_{ii}) + \psi(A_{ji})] \diamond_1 X_{ji} + (A_{ii} + A_{ji}) \diamond_1 \psi(X_{ji}). \end{aligned}$$

Therefore,

$$\psi(A_{ii} + A_{ji}) \diamond_1 X_{ji} + (A_{ii} + A_{ji}) \diamond_1 \psi(X_{ji}) = [\psi(A_{ii}) + \psi(A_{ji})] \diamond_1 X_{ji} + (A_{ii} + A_{ji}) \diamond_1 \psi(X_{ji}).$$

Hence, $K \diamond_1 X_{ji} = 0$ where $K = \psi(A_{ii} + A_{ji}) - \psi(A_{ii}) - \psi(A_{ji})$. So, $KX_{ji} = -X_{ji}K^*$ for all $X_{ji} \in \mathcal{A}_{ji}$. By using Lemma 2.2, we have

$$K = \alpha P_j - \bar{\alpha} P_i$$

for some $\alpha \in \mathbb{C}$. This implies

$$\psi(A_{ii} + A_{ji}) = \psi(A_{ii}) + \psi(A_{ji}) + \alpha P_j - \bar{\alpha} P_i.$$

Since $X_{jj} \diamond_1 A_{ji} = X_{jj} \diamond_1 (A_{ii} + A_{ji})$ for all $X_{jj} \in A_{jj}$ and $\psi(X_{jj}) \in \mathcal{A}_{jj}$, we can write

$$\begin{aligned} \psi(X_{jj}) \diamond_1 A_{ji} + X_{jj} \diamond_1 \psi(A_{ji}) &= \psi(X_{jj}) \diamond_1 (A_{ii} + A_{ji}) + X_{jj} \diamond_1 \psi(A_{ii} + A_{ji}) \\ &= \psi(X_{jj}) \diamond_1 A_{ji} + X_{jj} \diamond_1 \psi(A_{ii} + A_{ji}) \\ &= \psi(X_{jj}) \diamond_1 A_{ji} + X_{jj} \diamond_1 [\psi(A_{ii}) + \psi(A_{ji}) + \alpha P_j - \bar{\alpha} P_i] \\ &= \psi(X_{jj}) \diamond_1 A_{ji} + X_{jj} \diamond_1 \psi(A_{ji}) + X_{jj} \diamond_1 \alpha P_j. \end{aligned}$$

So, $X_{jj} \diamond_1 \alpha P_j = 0$. Thus, $\alpha X_{jj} = -\alpha X_{jj}^*$ for all $X_{jj} \in \mathcal{A}_{jj}$ and so $\alpha = 0$. Hence,

$$\psi(A_{ii} + A_{ji}) = \psi(A_{ii}) + \psi(A_{ji})$$

(d) From the fact $P_i \diamond_1 (A_{ij} + A_{ji}) = A_{ij} + A_{ji}$, we have

$$\psi(P_i)(A_{ij} + A_{ji}) + (A_{ij} + A_{ji})\psi(P_i) + P_i\psi(A_{ij} + A_{ji}) + \psi(A_{ij} + A_{ji})P_i = \psi(A_{ij} + A_{ji}).$$

Multiplying above equation by P_j from two sides, we have

$$P_j\psi(A_{ij} + A_{ji})P_j = 0.$$

Similarly, from $P_j \diamond_1 (A_{ij} + A_{ji}) = A_{ij} + A_{ji}$, we have $P_i\psi(A_{ij} + A_{ji})P_i = 0$.

On the other hand, from $(A_{ij} + A_{ji}) \diamond_1 P_i = A_{ji} \diamond_1 P_i$ we have

$$\psi(A_{ij} + A_{ji}) \diamond_1 P_i + (A_{ij} + A_{ji}) \diamond_1 \psi(P_i) = \psi(A_{ji}) \diamond_1 P_i + A_{ji} \diamond_1 \psi(P_i).$$

This implies that $K \diamond_1 P_i = 0$ where $K = \psi(A_{ij} + A_{ji}) - \psi(A_{ji})$. So, $KP_i + P_iK^* = 0$. Thus $P_jKP_i = 0$ and so $P_j\psi(A_{ij} + A_{ji})P_i = \psi(A_{ji})$.

Similarly, from $(A_{ij} + A_{ji}) \diamond_1 P_j = A_{ij} \diamond_1 P_j$, we can obtain $P_i\psi(A_{ij} + A_{ji})P_j = \psi(A_{ij})$.

These relations show that

$$\psi(A_{ij} + A_{ji}) = \psi(A_{ij}) + \psi(A_{ji}).$$

Step 5. For $1 \leq i \neq j \leq 2$, we have $\psi(\sum_{i,j=1,2} A_{ij}) = \psi(A_{ii}) + \psi(A_{ij}) + \psi(A_{ji}) + \psi(A_{jj})$.

First we show that

$$\psi(A_{ii} + A_{ij} + A_{ji}) = \psi(A_{ii}) + \psi(A_{ij}) + \psi(A_{ji}).$$

From $P_j \diamond_1 (A_{ii} + A_{ij} + A_{ji}) = P_j \diamond_1 (A_{ij} + A_{ji})$ and part (d) of Step 4 we have

$$\psi(P_j) \diamond_1 (A_{ii} + A_{ij} + A_{ji}) + P_j \diamond_1 \psi(A_{ii} + A_{ij} + A_{ji}) = \psi(P_j) \diamond_1 (A_{ij} + A_{ji}) + P_j \diamond_1 (\psi(A_{ij}) + \psi(A_{ji})).$$

So, $P_j \diamond_1 \psi(A_{ii} + A_{ij} + A_{ji}) = P_j \diamond_1 [\psi(A_{ij}) + \psi(A_{ji})]$.

Hence, $P_j \diamond_1 K = 0$ where $K = \psi(A_{ii} + A_{ij} + A_{ji}) - \psi(A_{ij}) - \psi(A_{ji})$. Then $P_jK + KP_j = 0$ and so, $P_jKP_j = P_iKP_j = P_jKP_i = 0$. Thus we have

$$P_j\psi(A_{ii} + A_{ij} + A_{ji})P_j = 0,$$

$$P_i\psi(A_{ii} + A_{ij} + A_{ji})P_j = \psi(A_{ij})$$

and

$$P_j\psi(A_{ii} + A_{ij} + A_{ji})P_i = \psi(A_{ji}).$$

On the other hand, from $(A_{ii} + A_{ij} + A_{ji}) \diamond_1 T_{ii} = (A_{ii} + A_{ji}) \diamond_1 T_{ii}$ and part (c) of Step 4 we have $L \diamond_1 T_{ii} = 0$ where $L = \psi(A_{ii} + A_{ij} + A_{ji}) - \psi(A_{ii}) - \psi(A_{ji})$. Thus, $LT_{ii} + T_{ii}L^* = 0$. By Lemma 2.1, the latter equation yields $P_iLP_i = \alpha P_i$ for some $\alpha \in \mathbb{C}$. Thus,

$$P_i\psi(A_{ii} + A_{ij} + A_{ji})P_i = \psi(A_{ii}) + \alpha P_i.$$

Hence, we obtain

$$\psi(A_{ii} + A_{ij} + A_{ji}) = \psi(A_{ii}) + \psi(A_{ij}) + \psi(A_{ji}) + \alpha P_i.$$

We will show that $\alpha = 0$. From above relation, for every $T_{ii} \in \mathcal{A}_{ii}$, there exists $\lambda \in \mathbb{C}$ such that

$$\begin{aligned} \psi[T_{ii} \diamond_1 (A_{ii} + A_{ij} + A_{ji})] &= \psi(T_{ii}A_{ii} + T_{ii}A_{ij} + A_{ii}T_{ii}^* + A_{ji}T_{ii}^*) \\ &= \psi(T_{ii}A_{ii} + A_{ii}T_{ii}^*) + \psi(T_{ii}A_{ij}) \\ &\quad + \psi(A_{ji}T_{ii}^*) + \lambda P_i. \end{aligned} \quad (2.15)$$

On the other hand,

$$\begin{aligned} \psi[T_{ii} \diamond_1 (A_{ii} + A_{ij} + A_{ji})] &= \psi(T_{ii}) \diamond_1 (A_{ii} + A_{ij} + A_{ji}) + T_{ii} \diamond_1 \psi(A_{ii} + A_{ij} + A_{ji}) \\ &= \psi(T_{ii}) \diamond_1 (A_{ii} + A_{ij} + A_{ji}) + T_{ii} \diamond_1 (\psi(A_{ii}) + \psi(A_{ij}) \\ &\quad + \psi(A_{ji}) + \alpha P_i) \\ &= \psi(T_{ii}) \diamond_1 A_{ii} + T_{ii} \diamond_1 \psi(A_{ii}) + \psi(T_{ii}) \diamond_1 A_{ij} + T_{ii} \diamond_1 \psi(A_{ij}) \\ &\quad + \psi(T_{ii}) \diamond_1 A_{ji} + T_{ii} \diamond_1 \psi(A_{ji}) + T_{ii} \diamond_1 \alpha P_i \\ &= \psi(T_{ii} \diamond_1 A_{ii}) + \psi(T_{ii} \diamond_1 A_{ij}) + \psi(T_{ii} \diamond_1 A_{ji}) + \alpha(T_{ii} + T_{ii}^*) \\ &= \psi(T_{ii}A_{ii} + A_{ii}T_{ii}^*) + \psi(T_{ii}A_{ij}) + \psi(A_{ji}T_{ii}^*) + \alpha(T_{ii} \\ &\quad + T_{ii}^*). \end{aligned}$$

Therefore, from relation (2.15) we have $\lambda P_i = \alpha(T_{ii} + T_{ii}^*)$ for every $T_{ii} \in \mathcal{A}_{ii}$.

Thus $\lambda = \alpha = 0$ and finally we have

$$\psi(A_{ii} + A_{ij} + A_{ji}) = \psi(A_{ii}) + \psi(A_{ij}) + \psi(A_{ji}). \quad (2.16)$$

Now, we prove

$$\psi(A_{ii} + A_{ij} + A_{ji} + A_{jj}) = \psi(A_{ii}) + \psi(A_{ij}) + \psi(A_{ji}) + \psi(A_{jj}).$$

Since $P_i \diamond_1 (\sum_{i,j=1,2} A_{ij}) = P_i \diamond_1 (A_{ii} + A_{ij} + A_{ji})$, we have

$$\psi(P_i \diamond_1 (\sum_{i,j=1,2} A_{ij})) + P_i \diamond_1 \psi(\sum_{i,j=1,2} A_{ij}) = \psi(P_i \diamond_1 (A_{ii} + A_{ij} + A_{ji})) + P_i \diamond_1 \psi(A_{ii} + A_{ij} + A_{ji}).$$

From relation (2.16) we have $P_i \diamond_1 M = 0$ where

$$M = \psi(\sum_{i,j=1,2} A_{ij}) - \psi(A_{ii}) - \psi(A_{ij}) - \psi(A_{ji}).$$

Hence, $P_i M + M P_i = 0$. This implies that $P_i M P_i = P_i M P_j = P_j M P_i = 0$.

Therefore,

$$\begin{aligned} P_i \psi(\sum_{i,j=1,2} A_{ij}) P_i &= \psi(A_{ii}), \\ P_i \psi(\sum_{i,j=1,2} A_{ij}) P_j &= \psi(A_{ij}) \end{aligned}$$

and

$$P_j \psi \left(\sum_{i,j=1,2} A_{ij} \right) P_i = \psi(A_{ji}).$$

Since $P_j \diamond_1 (\sum_{i,j=1,2} A_{ij}) = P_j \diamond_1 (A_{ij} + A_{ji} + A_{jj})$ by a similar method, we have

$$P_j \psi \left(\sum_{i,j=1,2} A_{ij} \right) P_j = \psi(A_{jj}).$$

Finally, we have

$$\psi \left(\sum_{i,j=1,2} A_{ij} \right) = \psi(A_{ii}) + \psi(A_{ij}) + \psi(A_{ji}) + \psi(A_{jj}).$$

Step 6. $\psi(A_{ij} + B_{ij}) = \psi(A_{ij}) + \psi(B_{ij})$ for every $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ such that $i, j = 1, 2$.

Let $i \neq j$, then $T_{ij} + T_{ij}^* = T_{ij} \diamond_1 P_j$, and so, by part (d) of Step 4, we have

$$\begin{aligned} \psi(T_{ij}) + \psi(T_{ij}^*) &= \psi(T_{ij}) \diamond_1 P_j + T_{ij} \diamond_1 \psi(P_j) \\ &= \psi(T_{ij}) + \psi(T_{ij})^* + T_{ij} \psi(P_j) + \psi(P_j) T_{ij}^*. \end{aligned}$$

Multiplying above equation by P_j from the right side, we have $T_{ij} \psi(P_j) = 0$ for every $T_{ij} \in \mathcal{A}_{ij}$. So, $\psi(P_j) = 0$ for $j = 1, 2$.

Let $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ such that $(i \neq j)$. Then,

$$A_{ij} + B_{ij} + A_{ij}^* + B_{ij} A_{ij}^* = (P_i + A_{ij}) \diamond_1 (P_j + B_{ij}),$$

and so, by Steps 4 and 5, we have

$$\begin{aligned} \psi(A_{ij} + B_{ij}) + \psi(A_{ij}^*) + \psi(B_{ij} A_{ij}^*) &= \psi(A_{ij} + B_{ij} + A_{ij}^* + B_{ij} A_{ij}^*) \\ &= \psi[(P_i + A_{ij}) \diamond_1 (P_j + B_{ij})] \\ &= \psi(P_i + A_{ij}) \diamond_1 (P_j + B_{ij}) \\ &\quad + (P_i + A_{ij}) \diamond_1 \psi(P_j + B_{ij}) \\ &= [\psi(P_i) + \psi(A_{ij})] \diamond_1 (P_j + B_{ij}) \\ &\quad + (P_i + A_{ij}) \diamond_1 [\psi(P_j) + \psi(B_{ij})] \\ &= \psi(A_{ij}) \diamond_1 (P_j + B_{ij}) + (P_i + A_{ij}) \diamond_1 \psi(B_{ij}) \\ &= \psi(A_{ij}) + \psi(B_{ij}) + \psi(A_{ij})^* + B_{ij} \psi(A_{ij})^* \\ &\quad + \psi(B_{ij}) A_{ij}^*. \end{aligned}$$

Multiplying by P_j from the right side implies that $\psi(A_{ij} + B_{ij}) = \psi(A_{ij}) + \psi(B_{ij})$ for every $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ such that $i \neq j$.

Let $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ and $T_{ij} \in \mathcal{A}_{ij}$. It follows from above relation that

$$\begin{aligned}
\psi[(A_{ii} + B_{ii}) \diamond_1 T_{ij}] &= \psi(A_{ii}T_{ij} + B_{ii}T_{ij}) \\
&= \psi(A_{ii}T_{ij}) + \psi(B_{ii}T_{ij}) \\
&= \psi(A_{ii} \diamond_1 T_{ij}) + \psi(B_{ii} \diamond_1 T_{ij}) \\
&= \psi(A_{ii}) \diamond_1 T_{ij} + A_{ii} \diamond_1 \psi(T_{ij}) + \psi(B_{ii}) \diamond_1 T_{ij} + B_{ii} \diamond_1 \psi(T_{ij}) \\
&= \psi(A_{ii})T_{ij} + A_{ii}\psi(T_{ij}) + \psi(B_{ii})T_{ij} + B_{ii}\psi(T_{ij}).
\end{aligned}$$

So,

$$\psi[(A_{ii} + B_{ii}) \diamond_1 T_{ij}] = \psi(A_{ii})T_{ij} + A_{ii}\psi(T_{ij}) + \psi(B_{ii})T_{ij} + B_{ii}\psi(T_{ij}).$$

On the other hand, since $\psi(A_{ii} + B_{ii}) \in \mathcal{A}_{ii}$ and above equation, we have

$$\begin{aligned}
\psi[(A_{ii} + B_{ii}) \diamond_1 T_{ij}] &= \psi(A_{ii} + B_{ii}) \diamond_1 T_{ij} + (A_{ii} + B_{ii}) \diamond_1 \psi(T_{ij}) \\
&= \psi(A_{ii} + B_{ii})T_{ij} + A_{ii}\psi(T_{ij}) + B_{ii}\psi(T_{ij}).
\end{aligned}$$

Hence, $[\psi(A_{ii} + B_{ii}) - \psi(A_{ii}) - \psi(B_{ii})]T_{ij} = 0$ for every $T_{ij} \in \mathcal{A}_{ij}$. This implies that $\psi(A_{ii} + B_{ii}) = \psi(A_{ii}) + \psi(B_{ii})$.

Step 7. ψ is additive and $*$ -preserving on \mathcal{A} .

Let $A = \sum_{i,j=1,2} A_{ij}$ and $B = \sum_{i,j=1,2} B_{ij}$ for every $A, B \in \mathcal{A}$, then from Steps 5 and 6 we have

$$\begin{aligned}
\psi(A + B) &= \psi\left(\sum_{i,j=1,2} A_{ij} + \sum_{i,j=1,2} B_{ij}\right) \\
&= \psi\left(\sum_{i,j=1,2} (A_{ij} + B_{ij})\right) \\
&= \sum_{i,j=1,2} \psi(A_{ij} + B_{ij}) \\
&= \sum_{i,j=1,2} \psi(A_{ij}) + \sum_{i,j=1,2} \psi(B_{ij}) \\
&= \psi\left(\sum_{i,j=1,2} A_{ij}\right) + \psi\left(\sum_{i,j=1,2} B_{ij}\right) = \psi(A) + \psi(B).
\end{aligned}$$

Then ψ is additive.

Now, we will prove ψ is $*$ -preserving. We showed in Step 6 that $\Phi(P_i) = 0$ for $i = 1, 2$. So, $\psi(I) = \Phi(P_1) + \Phi(P_2) = 0$. So,

$$\psi(A \diamond_1 I) = \psi(A) \diamond_1 I + A \diamond_1 \psi(I),$$

for all $A \in \mathcal{A}$, yields $\psi(A + A^*) = \psi(A) + \psi(A)^*$. Since ψ is additive we have $\psi(A) + \psi(A^*) = \psi(A) + \psi(A)^*$, for all $A \in \mathcal{A}$, and so $\psi(A^*) = \psi(A)^*$. Thus ψ is $*$ -preserving.

Step 8. $\psi(AB) = \psi(A)B + A\psi(B)$ for every $A, B \in \mathcal{A}$.

Here, we prove our Step by three cases.

Case 1. Let $A^* = -A$ (skew self-adjoint) and $B^* = B$.

By our Main Theorem assumption we have

$$\psi(A \diamond_1 B) = \psi(A) \diamond_1 B + A \diamond_1 \psi(B)$$

and

$$\psi(B \diamond_1 A) = \psi(B) \diamond_1 A + B \diamond_1 \psi(A).$$

By Step 7, we know ψ is $*$ -preserving, i.e., $\psi(T^*) = \psi(T)^*$ for every $T \in \mathcal{A}$.

So, from above relation we have the following

$$\psi(AB - BA) = \psi(A)B - B\psi(A) + A\psi(B) - \psi(B)A,$$

and

$$\psi(BA + AB) = \psi(B)A + A\psi(B) + B\psi(A) + \psi(A)B.$$

Adding these relations, by additivity of ψ , we have

$$\psi(AB) = \psi(A)B + A\psi(B) \quad (2.17)$$

for $A^* = -A$ and $B^* = B$. It means ψ is derivation for skew self-adjoint A and self-adjoint B .

Case 2. Let A and B be self-adjoint.

Before we prove ψ is derivation for self-adjoint operators, we need to show $\psi(iA) = i\psi(A)$, for all $A \in \mathcal{A}$. For this purpose we should verify $\psi(iI) = 0$.

We have $iT_{21} + iT_{21}^* = T_{21} \diamond_1 iP_1$ for all $T_{21} \in \mathcal{A}_{21}$, and so, by additivity of ψ and getting ψ of the latter equation, we have

$$\psi(iT_{21}) + \psi(iT_{21}^*) = i\psi(T_{21})P_1 + iP_1\psi(T_{21})^* + T_{21}\psi(iP_1) + \psi(iP_1)T_{21}^*.$$

Multiplying above equation by P_1 from the right side and also from $\psi(T_{21}) \in \mathcal{A}_{21}$, we have

$$\psi(iT_{21}) = i\psi(T_{21}) + T_{21}\psi(iP_1). \quad (2.18)$$

Let put iT_{21} instead of T_{21} in Equation (2.18) we have

$$-\psi(T_{21}) = i\psi(iT_{21}) + iT_{21}\psi(iP_1), \quad (2.19)$$

and multiply Equation (2.18) by i , we have

$$i\psi(iT_{21}) = -\psi(T_{21}) + iT_{21}\psi(iP_1). \quad (2.20)$$

Adding Equations (2.19) and (2.20) together, we have $2iT_{21}\psi(iP_1) = 0$, for all $T_{21} \in \mathcal{A}_{21}$. We obtain $\psi(iP_1) = 0$, since \mathcal{A}_{21} is prime.

By a similar way, we can obtain $\psi(iP_2) = 0$. Then, from additivity of ψ ,

$$\psi(iI) = \psi(iP_1) + \psi(iP_2) = 0.$$

Now, we are ready to show $\psi(iA) = i\psi(A)$, for all $A \in \mathcal{A}$.

By applying Equation (2.17) and the fact that $\psi(iI) = 0$, we have

$$\psi(iA) = \psi(iIA) = \psi(iI)A + iI\psi(A) = i\psi(A), \quad (2.21)$$

for all self-adjoint operators $A \in \mathcal{A}$. It is easy to see that we have $\psi(iA) = i\psi(A)$, for all $A \in \mathcal{A}$, since we can write as follow by additivity of ψ and Equation (2.21)

$$\begin{aligned} \psi(iA) &= \psi(i(\Re(A) + i\Im(A))) \\ &= \psi(i\Re(A) - \Im(A)) \\ &= \psi(i\Re(A)) - \psi(\Im(A)) \\ &= i[\psi(\Re(A)) + i\psi(\Im(A))] \\ &= i\psi(\Re(A) + i\Im(A)) = i\psi(A). \end{aligned}$$

So, we proved that

$$\psi(iA) = i\psi(A), \quad (2.22)$$

for all $A \in \mathcal{A}$.

Now, let get back to prove ψ is derivation for self-adjoint operators $A, B \in \mathcal{A}$. By Equation (2.1) we have the following

$$\psi(AB + BA) = \psi(A)B + B\psi(A) + A\psi(B) + \psi(B)A, \quad (2.23)$$

for all self-adjoint operators $A, B \in \mathcal{A}$.

On the other hand, by applying Equation (2.1) for iB and iA we have $\psi(iB \diamond_1 iA) = \psi(iB) \diamond_1 iA + iB \diamond_1 \psi(iA)$.

Hence, by Equation (2.21) we have

$$\begin{aligned} \psi(-BA + AB) &= \psi(iB)iA + iA\psi(iB)^* + iB\psi(iA) + \psi(iA)(iB)^* \\ &= -\psi(B)A + A\psi(B) - B\psi(A) + \psi(A)B. \end{aligned}$$

So,

$$\psi(-BA + AB) = -\psi(B)A + A\psi(B) - B\psi(A) + \psi(A)B, \quad (2.24)$$

for all self-adjoint $A, B \in \mathcal{A}$. Adding Equations (2.23) and (2.24) we have

$$\psi(AB) = \psi(A)B + A\psi(B) \quad (2.25)$$

for self adjoint operators in \mathcal{A} .

Case 3. Finally, we prove ψ is derivation for all A and B in \mathcal{A} . By Equation

(2.25) we have the following

$$\begin{aligned}
\psi(AB) &= \psi[(\Re(A) + i\Im(A))(\Re(B) + i\Im(B))] \\
&= \psi(\Re(A)\Re(B)) + i\psi(\Re(A)\Im(B)) + i\psi(\Im(A)\Re(B)) - \psi(\Im(A)\Im(B)) \\
&= \psi(\Re(A))\Re(B) + \Re(A)\psi(\Re(B)) + i\psi(\Re(A))\Im(B) + i\Re(A)\psi(\Im(B)) \\
&\quad + i\psi(\Im(A))\Re(B) + i\Im(A)\psi(\Re(B)) - \psi(\Im(A))\Im(B) - \Im(A)\psi(\Im(B)) \\
&= \psi(\Re(A))\Re(B) + \Re(A)\psi(\Re(B)) + i\psi(\Re(A))\Im(B) + i\Re(A)\psi(\Im(B)) \\
&\quad + i\psi(\Im(A))\Re(B) + i\Im(A)\psi(\Re(B)) + \psi(i\Im(A))i\Im(B) + i\Im(A)\psi(i\Im(B)) \\
&= \psi(\Re(A))[\Re(B) + i\Im(B)] + \Re(A)\psi[\Re(B) + i\Im(B)] \\
&\quad + i\psi(\Im(A))[\Re(B) + i\Im(B)] + i\Im(A)\psi[\Re(B) + i\Im(B)] \\
&= \psi(\Re(A))B + \Re(A)\psi(B) + i\psi(\Im(A))B + i\Im(A)\psi(B) \\
&= \psi(A)B + A\psi(B).
\end{aligned}$$

This completes the proof of main Theorem.

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